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AG II - Schemes and sheaves

Solutions – week 1

Exercise 1. (5) The universal property of $R' \otimes_R R''$ is that a R-algebra map out of this to an R-algebra S is the same as a pair of maps of R-algebras $R' \to S$ and $R \to S$. It is therefore the *coproduct* of R and R' in the category of R-algebras.

(7) $A \otimes_k B$ is to be interpreted as function on the product of the associated closed subspaces of $\mathbb{A}^n_k$ and $A \otimes_{k[x_1,\dots,x_n]} B$ as functions on their intersection in $\mathbb{A}^n_k$.

Exercise 2. We first expose a proof for sheaves of sets. Let (U_i) be an open cover of X.[1] Let $\varphi_i : \mathcal{F}_{U_i} \to \mathcal{G}_{U_i}$ be a collection of morphisms who agree on intersection. We show that it lifts uniquely to a morphism of presheaves $\mathcal{F} \to \mathcal{G}$.

Let V be any open of X. Consider $s \in \mathcal{F}(V)$. Using that $\mathcal{G}$ is a sheaf, that morphisms agree on intersections, and that φ_i is a morphism of presheaves for all i, we get that $(\varphi_{i,V\cap U_i}(s_{V\cap U_i}))$ lifts uniquely to an element of $\mathcal{G}(V)$ that we denote by $\varphi_V(s)$. We want to show that $(\varphi_V : \mathcal{F}(V) \to \mathcal{G}(V))$ is a morphism of presheaves. To see that, note that if $V' \subset V$ and $s \in \mathcal{F}(V)$,

$$\varphi_{V'}(s_{V'})_{|V'\cap U_i} \overset{\text{def. of } \varphi}{=} \varphi_{i,V'\cap U_i}(s_{V'\cap U_i})$$

$$\overset{\varphi \text{ is a morphism of presheaves}}{=} \varphi_{i,V\cap U_i}(s_{V\cap U_i})_{|V'\cap U_i} \overset{\text{def. of } \varphi}{=} \varphi_V(s)_{|V'\cap U_i}$$

so both $\varphi_{V'}(s_{V'})$ and $\varphi_V(s)_{V'}$ restrict on $V' \cap U_i$ to the same element. As $\mathcal{G}$ is a sheaf, the desired equality follows. Note that for any $V \subset U_i$ we see by definition that $\varphi_V = \varphi_{i,V}$. This shows the existence of the lift.

As for the unicity note that value on $s \in \mathcal{F}(V)$ of a lift φ' necessarily restricts to $(\varphi_{i,V\cap U_i}(s_{V\cap U_i}))$. Therefore the uniqueness follows from the uniqueness in the sheaf property of $\mathcal{F}$.[2]

We answer now a question asked during TA sessions : can we do this with sheaves with value in an arbitrary category $\mathcal{C}$? The answer is yes and we will do some preliminary definitions. Note that in the above proof there is essentialy three steps: one commutative diagram to show the existence, one to show that this defines a natural transformation, and one argument for the unicity. The proof below is the same pattern.

[1] This case will suffice; for a general open V we can apply the reasoning to $X = V$ and $\mathcal{F} = \mathcal{F}_{|V}$ and $\mathcal{G} = \mathcal{G}_{|V}$.

[2] If one now wants to show a similar statement for sheaves of abelian groups/rings/etc. one can now argue that to verify that a morphism of presheaves of sets is a morphism of presheaves of abelian groups/rings/etc. it suffices to check it at stalks/locally, which will hold because by construction it will already hold locally.

Let $\mathcal{C}$ be a complete category. A sheaf $\mathcal{F}$ on X with values in $\mathcal{C}$ is a presheaf such that for any open U of X and open covering (U_i) of U, the following[3]

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{ij})$$

is an equalizer diagram. We denote by $\mathrm{Sh}_{\mathcal{C}}(X)$ the full subcategory of $\mathrm{Psh}_{\mathcal{C}}(X) = \mathrm{Fun}(\mathrm{Ouv}(X)^{op}, \mathcal{C})$ consisting of sheaves with values in $\mathcal{C}$.
Now we define the Set-valued presheaf

$$U \mapsto \mathrm{Hom}_{\mathrm{Sh}_{\mathcal{C}}(U)}(\mathcal{F}_U, \mathcal{G}_U)$$

Now we want to show that this pre-sheaf is a sheaf, if we make the hypothesis that $\mathcal{G}$ is a sheaf. To show this, take $(U_i)_{i\in I}$ an open cover of $U \in \mathrm{Ouv}\, X$ and a collection of natural transformations

$$(\alpha^i \colon \mathcal{F}_{U_i} \to \mathcal{G}_{U_i})_{i\in I}$$

such that for all $i, j \in I$ and $W \subset U_{ij}$

$$(\alpha^i_W \colon \mathcal{F}(W) \to \mathcal{G}(W)) = (\alpha^j_W \colon \mathcal{F}(W) \to \mathcal{G}(W)). \tag{1}$$

We need to show that there is a unique natural transformation $\widehat{\alpha} \colon \mathcal{F}_U \to \mathcal{G}_U$ such that restricting this natural transformation to a U_i gives α_i.
Let $V \subset U$ be open. By the universal property of the product, let :

$$\beta_V : \mathcal{F}(V) \to \prod_{i\in I} \mathcal{G}(V \cap U_i)$$

induced by

$$\mathcal{F}(V) \to \mathcal{F}(V \cap U_i) \xrightarrow{\alpha^i_{V\cap U_i}} \mathcal{G}(V \cap U_i).$$

Now we want to consider $\widehat{\alpha}_V \colon \mathcal{F}(V) \to \mathcal{G}(V)$ the unique morphism who would be given the universal property of the following equalizer (because $\mathcal{G}$ is a sheaf) for the cover of V being $(V \cap U_i)_i$. Note that if $V \subset U_i$, by construction, we will have $\widehat{\alpha}_V = \alpha^i_V$.

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[d, dashed, "\widehat{\alpha}_V"'] \arrow[dr, "\beta_V"] & & \\
\mathcal{G}(V) \arrow[r] & \prod_{i\in I} \mathcal{G}(V \cap U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} \mathcal{G}(V \cap U_{ij})
\end{tikzcd}$$

To see that this works, we need to show that β_V commutes indeed in this diagram.
This holds, because of the commutative the diagram below, who commutes because $\mathcal{F}$ and $\mathcal{G}$ are functors, that α^i, α^j are natural transformations and that using (1) we have $\alpha^i_{V\cap U_{ij}} = \alpha^j_{V\cap U_{ij}}$.

[3] with the two maps being on component (i, j) once $\prod_k \mathcal{F}(U_k) \to \mathcal{F}(U_i) \to \mathcal{F}(U_{ij})$ and $\prod_k \mathcal{F}(U_k) \to \mathcal{F}(U_j) \to \mathcal{F}(U_{ij})$ the other time

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d] & \mathcal{F}(V\cap U_i) \arrow[d] \arrow[dr, "\alpha^i_{V\cap U_i}"] & & \\
\mathcal{F}(V\cap U_j) \arrow[r] \arrow[dr, "\alpha^j_{V\cap U_j}"'] & \mathcal{F}(V\cap U_{ij}) \arrow[ddrr, "\alpha^i_{V\cap U_{ij}}=\alpha^j_{V\cap U_{ij}}"] & \mathcal{G}(V\cap U_i) \arrow[ddr, bend left] & \\
& \mathcal{G}(V\cap U_j) \arrow[drr, bend right] & & \\
& & & \mathcal{G}(V\cap U_{ij})
\end{tikzcd}$$

So $\widehat{\alpha}_V\colon \mathcal{F}(V) \to \mathcal{G}(V)$ is indeed well defined.
We claim that $(\widehat{\alpha}_V : \mathcal{F}(V) \to \mathcal{G}(V))_{V\subset U}$ is a natural transformation lifting the collection above.
We show that $\widehat{\alpha}$ is natural. This mean we have to show that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d, "\widehat{\alpha}_V"'] & \mathcal{F}(V') \arrow[d, "\widehat{\alpha}_{V'}"] \\
\mathcal{G}(V) \arrow[r] & \mathcal{G}(V')
\end{tikzcd}$$

By the universal property of the equalizer (using again that $\mathcal{G}$ is a sheaf), it amounts to prove the commutativity of,

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d, "\widehat{\alpha}_V"'] & \mathcal{F}(V') \arrow[r, "\widehat{\alpha}_{V'}"] & \mathcal{G}(V') \arrow[d] \\
\mathcal{G}(V) \arrow[r] & \mathcal{G}(V') \arrow[r] & \prod_i \mathcal{G}(V'\cap U_i)
\end{tikzcd}$$

So using the universal property of the product, we need only to verify that for every i :

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d, "\widehat{\alpha}_V"'] & \mathcal{F}(V') \arrow[r, "\widehat{\alpha}_{V'}"] & \mathcal{G}(V') \arrow[r] & \prod_i \mathcal{G}(V'\cap U_i) \arrow[d] \\
\mathcal{G}(V) \arrow[r] & \mathcal{G}(V') \arrow[r] & \prod_i \mathcal{G}(V'\cap U_i) \arrow[r] & \mathcal{G}(V'\cap U_i)
\end{tikzcd}$$

commutes. But this holds because we can insert commutating diagrams inside the diagram above in the following way :

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[dr] \arrow[ddd, "\widehat{\alpha}_V"'] & \mathcal{F}(V') \arrow[r, "\widehat{\alpha}_{V'}"] \arrow[dr] & \mathcal{G}(V') \arrow[r] & \prod_i \mathcal{G}(V'\cap U_i) \arrow[ddd] \\
& \mathcal{F}(V\cap U_i) \arrow[r] \arrow[dr, "\alpha^i_{V\cap U_i}"'] & \mathcal{F}(V'\cap U_i) \arrow[ddr, "\alpha^i_{V'\cap U_i}"] & \\
& & \mathcal{G}(V\cap U_i) \arrow[dr] & \\
\mathcal{G}(V) \arrow[r] \arrow[uurr] & \mathcal{G}(V') \arrow[r] & \prod_i \mathcal{G}(V'\cap U_i) \arrow[r] & \mathcal{G}(V'\cap U_i)
\end{tikzcd}$$

The intermediate diagrams commute because of the functoriality of $\mathcal{F}$ and $\mathcal{G}$, the naturality of α^i and the definition of $\widehat{\alpha}$.
The unicity of the lift is left to show. Suppose that $\widehat{\alpha'}$ is a lift. Then for any V, and $i \in I$ we have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V \cap U_i) \\ {\scriptstyle \widehat{\alpha'}_V}\big\downarrow & & \big\downarrow{\scriptstyle \alpha^i_{V \cap U_i}} \\ \mathcal{G}(V) & \longrightarrow & \mathcal{G}(V \cap U_i) \end{array}$$

Therefore we see that by universal property of $\mathcal{G}(V)$ as an equalizer with respect to the sheaf property and the cover $(U_i \cap V)_i$ of V that $\widehat{\alpha'}_V = \widehat{\alpha}_V$.

Exercise 3. Let S be a set and X a topological space. In what follows we prove that on a connected open subspace U the canonical map $S \to \underline{S}(U)$ is a bijection. We use the following description

$$\underline{S}(U) = \{(s_x) \in \prod_{x \in X} S \mid \forall x \in X \quad \exists U \ni x \quad \forall y, y' \in U \quad s_y = s_{y'}\}$$

and the natural map $S \to \underline{S}(U)$ being the diagonal. Let $(t_x) \in \underline{S}(U)$. Fix $y \in U$ (connected implies non empty). Now note that

$$V_1 = \{x \in U \mid t_x = t_y\} \quad V_2 = \{x \in U \mid t_x \neq t_y\}$$

form a disjoint decomposition of U into open subspaces. As U is connected and $y \in V_1$ we get $V_2 = \emptyset$ and the claim follows.
Now, as any subset U of the real line is a disjoint union of connected open subsets (which is also true for any locally connected space), we get that $\underline{\mathbb{Q}}(U) = \prod_{\pi_0(U)} \mathbb{Q}$ using the sheaf property. This vector space is finite dimensional when U has finitely many connected components and the dimension is then equal to $\pi_0(U)$.

Exercise 4. (2) Everything in what follows works for a presheaf. Note first of all that any $s \in \mathcal{F}(V)$ the map $\widehat{s}\colon V \to |\mathcal{F}|$ defined by $x \mapsto s_x$ is a section of $p : |\mathcal{F}| \to X$. Note also that

$$\widehat{s}(V) = \{s_x \mid x \in V\}$$

is open. Indeed, we need to show by definition of the topology that for any V' open and $t \in \mathcal{F}(V')$

$$\widehat{t}^{-1}(\widehat{s}(V)) = \{x \in V \cap V' \mid s_x = t_x\}$$

is open. This follows from the following lemma about directed colimits.[4]

Lemma. *Let (A_i) be a directed system of sets and $\varinjlim_i A_i$ the colimit. If $a_i \in A_i$ and $a_j \in A_j$ coincide in the colimit, then there exists k with $i \to k$ and $j \to k$ with the image of a_i and a_j being the same in A_k.*

[4]As forgetful functors to sets from abelian groups or rings commute with directed colimits, this lemma also applies to directed colimits of abelian groups, rings.

Proof. One checks that the colimit is given by the quotient of $\bigsqcup_i A_i$ by the relation $(a_i \in A_i) \sim (a_j \in A_j)$ if and only if there exist $i \to k$ and $j \to k$ with a_i and a_k identified in A_k. Once this understood, the lemma follows. □

Now, it follows that $p : |\mathcal{F}| \to X$ is continuous. Indeed for an open set U of X we have

$$p^{-1}(U) = \bigsqcup_{(s,V), s \in \mathcal{F}(V)} \widehat{s}(V).$$

Also, we see that for any open V and $s \in \mathcal{F}(V)$ we have $p_{|\widehat{s}(V)} \widehat{s} = \mathrm{id}_V$ and $\widehat{s} p_{|\widehat{s}(V)} = \mathrm{id}_{\widehat{s}(V)}$. Therefore p is a local homeomorphism.

Remark. We have a natural isomorphism between $\mathcal{F}_p \to \mathcal{F}^+$. (Here $\mathcal{F}_p$ denotes the sheaf of sections of $p : |\mathcal{F}| \to X$.)

Exercise 5. To show that $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism at stalks, we proceed as follows. Note that for any open $U \ni x$ the following projection map

$$\mathcal{F}^+(U) \subset \prod_{x \in U} \mathcal{F}_x \to \mathcal{F}_x$$

will pass to the colimit $(\mathcal{F}^+)_x \to \mathcal{F}_x$. One immediately checks that this is an inverse to the induced map at stalks from $\mathcal{F} \to \mathcal{F}^+$.
For (2) and "not injective" we can take the presheaf on $\mathbb{R}$ with value $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{R}$ and 0 for any other open.
For "not surjective", take $\mathbb{R}$ and the sheafification of any non-zero abelian group. See "constant" sheaf exercise 3.

Exercise 6.

(1) Note that $e : [0, \frac{3}{2}] \to S^1$ is a local homeomorphism. We claim that the natural evaluation map

$$(\mathcal{F}_e)_z \xrightarrow{\mathrm{ev}_z} e^{-1}(z)$$

is a bijection.[5] Let $x \in e^{-1}(z)$. Let $U \ni Z$ such that $e_{|U}$ is an homeomorphism. Then $e_{|U}^{-1}(z) = x$. This shows surjectivity. If s, t are sections on say $V \ni Z$ and $V' \ni z$ which have the same value on z, say x, then take an open $U \ni x$ such that $e_{|U}$ is an homoemoprhism and $e(U) \subset V \cap V'$. Then $s_{|e(U)}$ and $t_{|e(U)}$ are both the unique inverse to $e_{|U}$. This shows the injectivity.

(2) We show that $\mathcal{O}_z$ is a local $\mathbb{R}$-algebra. We claim that the ideal

$$\{f \in \mathcal{O}_z \mid f(z) = 0\}$$

is the unique maximal ideal. To this end, it suffices to show that the complement consits of the invertible elements. If $f(z) \neq 0$, then there exists a neighbourghood of z where f never vanishes. Therefore $\frac{1}{f}$ is a well defined multiplicative inverse in the stalk.

Some setup and notations for the rest of the exercise.

(a) To avoid confusion, we write the complex number $e(0) = e(1) = 1 \in S^1$ by u.

[5]Note that the following argument holds true for any local homeomorphism $e : X \to Y$.

(b) Denote by $e\colon [0,1] \to S^1$ the quotient map given by $\exp(2\pi i -)$.
(c) The quotient map $p\colon [0,1] \times \mathbb{R} \to M$ gives an homeomorphism

$$p\colon (0,1) \times \mathbb{R} \to \pi^{-1}(S^1 \setminus u).$$

(d) The quotient map $p\colon [0,1] \times \mathbb{R} \to M$ gives an homeomorphism

$$p\colon [0, \frac{1}{2}] \times \mathbb{R} \to \pi^{-1}(S^1_{\geq 0}),$$

where $S^1_{\geq 0}$ denotes the points of the circle with imaginary part positive or zero.
(e) The quotient map $p\colon [0,1] \times \mathbb{R} \to M$ gives an homeomorphism

$$p\colon [\frac{1}{2}, 1] \times \mathbb{R} \to \pi^{-1}(S^1_{\leq 0}),$$

where $S^1_{\leq 0}$ denotes the points of the circle with imaginary part negative or zero.

Let $s \in \mathcal{L}(U)$ be a section. We define a continuous map $\alpha_s\colon e^{-1}(U) \to \mathbb{R}$ such that

$$s(e(t)) = [e(t), \alpha_s(t)].$$

For $t \neq 0, 1$, we define $\alpha_s(t)$ to be the second component of $p^{-1}(s(e(t)))$, by (c) above. When $t = 0$ and $t = 1$, we extend by continuity and the same method using the points (d) and (e) respectively. Note that

$$\alpha_s(0) = -\alpha_s(1)$$

because $s(u) = [0, \alpha_s(0)] = [1, \alpha_s(1)]$.

(3) We define a module structure. We explain how to define the multiplication by scalars, the others operations being defined similarly. Let U be any open of M. Let $f \in \mathcal{O}(U)$ and $s \in \mathcal{L}(U)$. We define $f \cdot s$ as follows. We pass to the quotient map $e\colon [0,1] \to S^1$, the following continuous map $[0,1] \to M$

$$t \mapsto [t, f(e(t))\alpha_s(t))].$$

To show that it passes to the quotient we have to show that it agrees on $t = 0$ and $t = 1$. But as

$$f(u)\alpha_s(0) = f(u)(-\alpha_s(1)) = -f(u)\alpha_s(1),$$

this follows from the quotient relation of the Möbius band.

The zero element is the section $s_0 : S^1 \to M$, $s_0(e(t)) = [t, 0]$.

One continue similarly to define the rest of the structure. The key is that the "gluing of the quotient" $(-1)\colon \mathbb{R} \to \mathbb{R}$ is an automorphism of $\mathbb{R}$-modules so that we can "lift" calculations to pointwise calculations in $[0,1] \times \mathbb{R}$. That's why we put the emphasis on that in the above calculation.

(4) For any section $s \in \mathcal{L}(U)$ we have the unique map

$$\mathcal{O}_{|U} \to \mathcal{L}_{|U}$$

that respects the module structure on each open subset of U and sends 1 to s. We claim that if s vanishes nowhere, then this map is an isomorphism. To prove that, we suppose that s vanishes nowhere, and construct an homeomorphism over U

$$\psi_s\colon \pi^{-1}(U) \to U \times \mathbb{R}$$

defined by $[t, \lambda] \mapsto (e(t), \frac{\lambda}{\alpha_s(t)})$. This is well defined by non-vanishing. The inverse is given by $(z, \lambda) \mapsto (\lambda \cdot s)(z)$, where $\cdot$ designates the module structure defined above. Now

$$pr_2\psi_s(-)\colon \mathcal{L}_{|U} \to \mathcal{O}_{|U}$$

gives an inverse to the above map.

We are now left to prove that on any open subset missing a point U, there exist a non-vanishing section. But whenever a point is missing, say $e(t_0) \notin U$ for some $t_0 \in [0, 1)$ then we can define the section $U \to M$ by

$$e(t) \mapsto \begin{cases} [t, 1] & t < t_0 \\ [t, -1] & t > t_0 \end{cases}$$

which vanishes nowhere.

(5) Let $s \in \mathcal{L}(S^1)$. By the intermediate value theorem $\alpha_s\colon [0, 1] \to \mathbb{R}$ necessarily vanishes because $\alpha_s(0) = -\alpha_s(1)$.

(6) Note that a section $s \in \mathcal{L}(U)$ vanishes at $z = e(t)$ in the sense that $s(z) = [t, 0]$ if and only if $s_z \in \mathfrak{m}_z\mathcal{L}_z$. Note that $1 \in \mathcal{O}(S^1)$ vanishes on no point. By contradiction, the image of 1 by an isomorphism $\mathcal{O} \cong \mathcal{L}$ would not vanish at any stalk, in contradiction with the previous point.